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N-Loop Treatment of Overlapping Diagrams by the Implicit Regularization Technique

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We show how the Implicit Regularization Technique (IRT) can be used for the perturbative renormalization of a simple field theoretical model generally used as a test theory for new techniques. While IRT has been applied successfully in many problems involving symmetry-breaking anomalies and nonabelian gauge groups, all at one-loop level, this is the first attempt at a generalization of the technique for perturbative renormalization. We show that the overlapping divergent loops can be given a completely algebraic treatment. We display the connection between renormalization and counterterms in the Lagrangian. The algebraic advantages make IRT worth studying for perturbative renormalization of gauge theories.

KEY WORDS: renormalization; regularization; *N*-loop treatment.

1. INTRODUCTION

Quantum field theoretical predictions of physical quantities should in principle be independent of the particular scheme used to renormalize the theory. The renormalization program gets rid of the singularities by redefining the parameters in the Lagrangian in a consistent way for a renormalizable model. In this process we must also make sure that the relevant symmetries of the underlying theory are preserved and must therefore avoid the appearance of spurious anomalies, which otherwise would have to be controlled order by order in perturbation theory by imposing symmetry restoring constraint equations.

As for the existing regularization schemes, while for the theories with low symmetry content nearly all regulators do a good job, this is not the case for most theories of particle interactions in which gauge symmetry, supersymmetry (SUSY), and the like play a fundamental role. Dimensional Regularization (DR) (Bollini and Giambiagi, 1972; 't Hooft and Veltman, 1972a,b) is an efficient and pragmatical

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method that explicitly preserves gauge symmetry. However in the presence of dimension specific objects such as γ^5 matrices, a suitable generalization of the Dirac algebra must be constructed to be compatible with analytical continuation on the space-time dimension. This is the case for the Electroweak sector of the Standard Model. Since chiral symmetry is broken in this case, the corresponding Ward–Slavnov–Taylor identities must be imposed order by order, which makes the computations beyond the one-loop order very hard.

For SUSY theories, due to the fact that the equality between Bose and Fermi degrees of freedom only holds for specific values of the space-time dimension, SUSY is broken in DR. A naive scheme (Dimensional Reduction) in which the field components are left unchanged while the loop integrals are performed in d dimensions can be shown to be inconsistent, see Siegl (1980). Similar problems arise in Chern–Simons field theories in which the Levi–Civita tensor is the three-dimensional analog of the γ^5 matrix (Carrington *et al.*, 1999; Chaichian and Chen, 1998; Giavarini and Martin, 1992; Martin, 1990).

A particularly interesting regularization independent framework is the Differential Renormalization Program pioneered by Freedman *et al.* (1992). The basic idea of this scheme is that renormalization comes from the fact that products of propagators must be extended to distributions so that a Fourier transform is well defined. Working in (Euclidean) coordinate space one writes the amplitude as a derivative of a distribution less divergent at coincident points. The derivatives are understood in the sense of distribution theory, i.e. acting formally by parts. The amplitudes written in this way are identical to the bare ones for separate points but behave well at coincident points. An intrinsic arbitrary scale appears in this process which is used as a Callan–Symanzik renormalization group parameter. The advantage of this method is that it works in integer space-time dimension, and it has been shown to yield satisfactory results where it was tested (del Águila *et al.*, 1997, 1998, 1999; del Águila and Pérez-Victoria, 1998; Hahn and Pérez-Victoria, 1999; Pérez-Victoria, 1998). However no general procedure using Differential Renormalization beyond one-loop order, such that gauge invariance is automatic, has been constructed yet.

Recently an essentially regularization independent procedure has been advanced (see Battistel, 1999; Battistel, *et al.*, 1998; Brizola *et al.*, 1999, 2002; Sampaio *et al.*, 2002; Scarpelli *et al.*, 2001). It presents the same consistency as Differential Regularization, working however in momentum space. It has been recently shown that the method respects both abelian and nonabelian symmetries at a few loops level and the infrared divergence can be treated by the same procedure constructed for ultraviolet divergences, without any modification, as has been shown for the gluon self-energy (Sampaio, *et al.*, 2002). All these results indicate that the method deserves further investigation, for example, the question of n-loops and perturbative renormalization in a theory. Encouraged by the previous results,

and aware of the difficulties of setting up a consistent scheme for perturbative renormalization where there are already so many of them so well established, for the sake of a complete evaluation of the Implicit Regularization Technique, we take one step in this direction in the present contribution. Since this is a major task, already pursued by several sophisticated schemes, we start by investigating a simple but nontrivial theory (involving, e.g. overlapping divergences) to test our method further. If it succeeds it will be a most important next step to construct a perturbative *n*-loop renormalization scheme in the spirit and consistency of e.g. BPHZ and others. For our present purposes we study specifically the ϕ_4^3 theory, to *n*-loop order.

The IRT is essentially regularization independent in the sense that a specific regulator needs never be used. A convenient identity at level of the integrand enables us to rewrite the amplitude as a sum of three types of contributions namely local divergences (basic divergent integrals which characterize the divergent structure of theory), nonlocal divergences typical of divergent substructures contributions), and finite contributions. The local divergences obtained in this way are equivalent to those obtained by performing a Taylor expansion (like in BPHZ) only for primitive diagrams, with no subdivergences. In the case subdivergences are present some more subtle differences between our method and BPHZ should be pointed out: a) Taylor expansions modify the original Feynman integrand; this procedure may therefore violate symmetries. We circumvent this problem by means of using a mathematical identity, preserving thus the original content of the original Feynman amplitude. b) In order to classify the subdivergences, BPHZ uses graphic representation (forest formula). In our procedure we identify, in terms of integrals, divergences which occurred at lower loop orders. This avoids in particular the complicated topological graphs structures and substitutes this step by an algebraic procedure, (at least in this simple case) from which the counterterms appear in a natural systematic way. Specific examples are given in the text. Moreover, just as in Differential Renormalization, arbitrary local terms can be duly parametrized and properly adjusted on physical grounds. This is particularly important for finite renormalization in order to clear the calculation of regularization ambiguities. Finally our framework lives in the integer space-time dimension which avoids well-known problems with dimension-specific theories.

2. RENORMALIZATION BY THE IMPLICIT REGULARIZATION TECHNIQUE

In this section we extent a technique first designed for one (Battistel, 1999; Battistel *et al.*, 1998; Scarpelli *et al.*, 2001) and two-loop calculations (Brizola *et al.*, 1999) to prove *nth* order renormalizability.

In order to illustrate the procedure, consider the following divergent amplitude, typical of one-loop order:

$$\int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+p)^2 - m^2](k^2 - m^2)}. \tag{1}$$

The symbol Λ under the integral sign presupposes, as discussed, an implicit regularization. Now, in order to separate the logarithmic divergence from the finite part, we use the following identity in the factor involving the external momentum p:

$$\frac{1}{[(k+p)^2 - m^2]} = \sum_{j=0}^{N} \frac{(-1)^j (p^2 + 2p\cdot k)^j}{(k^2 - m^2)^{j+1}} + \frac{(-1)^{N+1}(p^2 + 2p\cdot k)^{N+1}}{(k^2 - m^2)^{N+1}[(k+p)^2 - m^2]}. \tag{2}$$

In the above expression N is chosen so that the last term is finite under integration over k. Notice also that in the first term in Eq. (2), the external momentum appears only in the numerator and thus after integration it can yield at most polynomials in p multiplied by divergences. For our present example we need $N = 0$, since we are dealing with a logarithmic divergence. We can rewrite (1) using (2) as

$$I + \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} - \int \frac{d^4k}{(2\pi)^4} \frac{p^2 + 2p\cdot k}{[(k+p)^2 - m^2](k^2 - m^2)^2}. \tag{3}$$

Now only the first of these two integrals is divergent. The others can be easily integrated out to yield

$$I = I_{log}(m^2) - \frac{i}{(4\pi)^2} Z_0(m^2, p^2) \tag{4}$$

where

$$I_{log}(m^2) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \tag{5}$$

and

$$Z_0(m^2, p^2) = \int_0^1 dz \ln\left(\frac{p^2 z(1-z) - m^2}{-m^2}\right). \tag{6}$$

Note that, since no explicit form for the regulator has been used, one can make immediate contact with other regularizations. Details of calculations of several one-loop amplitudes and their associated Ward identities by using this method can be found in Scarpelli *et al.*, 2001.

By convenience we divide the diagrams which contribute to a given order in two classes: the first which does not contain diagrams which possess two-point functions as subdivergences and in the second class those which do.

Let us start with the first class of diagrams. To show how the procedure works it is enough to consider a general Feynman amplitude with one external momentum p, one coupling constant λ, and one mass parameter m. We work in the 4-dimensional space-time although the generalization to any integer dimension is straightforward. We denote by q a sum of internal momenta k_i. The amplitude in question can always be written as

$$\Gamma = \prod_{i=0}^{n} \int_{\Lambda} \frac{d^4 k_i}{(2\pi)^4} R(p, q, m, \lambda) \left[\prod_{j=1}^{l} f_j(p, q_j, m^2) \right] \tag{7}$$

where

$$f_j(p, q_j, m^2) = \frac{1}{[(p - q_j)^2 - m^2]} \tag{8}$$

and

$$l = \text{number of } f \text{ structures}$$

$$n = \text{number of loops.}$$

Note that we have explicitly separated the terms involving the external momentum in the denominator, from which nonlocal divergent contributions can arise after integration over the internal momenta. The structure $R(p, q, m, \lambda)$ contains all other ingredients of the amplitude such as coupling constants, results of Dirac traces, and so on.

For simplicity we adopt the following notation

$$\Gamma = (\Pi R)(\Pi f) \tag{9}$$

where

$$(\Pi R) = \prod_{i=1}^{n} \int_{\Lambda} \frac{d^4 k_i}{(2\pi)^4} R(p, q, m, \lambda) \tag{10}$$

and

$$(\Pi f) = \prod_{j=1}^{l} f_j(p, q_j, m^2). \tag{11}$$

As discussed before the source of all possible troubles in the renormalization process will arise from the structure (Πf). Our method focus attention on these structures. In order to clearly separate finite, local divergences (whose dependence on the external momenta is only a polynomial) from the nonlocal divergences we use a strategy which is completely based on the identity (2).

We define the operator T^D which acts on each structure f in the following way

$$T^0 f = \frac{1}{q_j^2 - m^2} + \frac{2p \cdot q_j - p^2}{\left(q_j^2 - m^2\right)} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\} \tag{12}$$

$$T^1 f = \frac{1}{q_j^2 - m^2} + \frac{(2p \cdot q_j - p^2)}{\left(q_j^2 - m^2\right)^2} + \frac{(2p \cdot q_j - p^2)^2}{\left(q_j^2 - m^2\right)^2} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\} \tag{13}$$

$$T^2 f = \frac{1}{q_j^2 - m^2} + \frac{(2p \cdot q_j - p^2)}{\left(q_j^2 - m^2\right)^2} + \frac{(2p \cdot q_j - p^2)^2}{\left(q_j^2 - m^2\right)^3} + \frac{(2p \cdot q_j - p^2)^3}{\left(q_j^2 - m^2\right)^3} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\}. \tag{14}$$

Note that the action of the operator T^D is equivalent to a Taylor expansion around zero external momentum where the first terms are kept and the rest of the series is resumed, yielding thus a convenient identity. Note also that the degree of divergence of the various terms is decreasing.

The procedure we have in mind consists of applying the operation, in a particular amplitude with superficial degree of divergence D, to each function f_j

$$T^D \Gamma = (\Pi R) \prod_{j=1}^{l} T_j^D f_j(p, q_j, m^2). \tag{15}$$

The result of the operation will always have the form

$$T^D f(p, q, m^2) = f^{\text{div}}(p, q, m^2) + f^{\text{fin}}(p, q, m^2). \tag{16}$$

We define

$$f^{\text{div}}(p, q, m^2) = \sum_{i=0}^{D} f^i(p, q, m^2). \tag{17}$$

Let us exemplify. Take a quadratically divergent amplitude. To each contribution of the form

$$\frac{1}{(p - q_j)^2 - m^2}$$

we associate

$$f^0(q, m^2) = \frac{1}{q^2 - m^2} \tag{18}$$

$$f^1(p, q, m^2) = \frac{2p \cdot q - p^2}{(q^2 - m^2)^2} \tag{19}$$

$$f^2(p,q,m^2) = \frac{(2p\cdot q)^2}{(q^2-m^2)^2} \tag{20}$$

and

$$f^{\text{fin}}(p,q,m^2) = \frac{p^4 - 4p^2(p\cdot q)}{(q^2-m^2)^3} + \frac{(2p\cdot q - p^2)^3}{(q^2-m^2)^3[(p-q_j)^2-m^2]}. \tag{21}$$

The definitions (18), (19), (20), and (21) are not unique. It is simply convenient for our purposes. Using these we rewrite the amplitude as a sum of various contributions. According to our notation

$$T^D\Gamma = (\Pi R)\prod_{j=1}^{l}\left[\, f_j^{\text{div}}(p,q,m^2) + f_j^{\text{fin}}(p,q,m^2)\right]. \tag{22}$$

In this way we can identify three distinct contributions for the amplitude

$$T^D\Gamma = \Gamma^1_{\text{fin}} + \Gamma_{\text{local}} + \Gamma_{\text{nonlocal}} \tag{23}$$

where

$$\Gamma^1_{\text{fin}} = (\Pi R)\prod_{j=1}^{l} f_j^{\text{fin}}(p,q,m^2). \tag{24}$$

The second contribution contains only local divergences and, for some particular (ΠR) structures, it can contain finite contributions too. It is identified as

$$\begin{aligned}\Gamma_{\text{local}} &= (\Pi R)\prod_{j=1}^{l} f_j^{\text{div}}(p,q,m^2)\\ &= \Gamma^2_{\text{fin}} + \Gamma^{\text{div}}_{\text{local}}.\end{aligned} \tag{25}$$

These local divergences correspond to counterterms which are characteristic of the order we are renormalizing. For example, they can have the form

$$\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2} + p^2 I_{\log}(m^2) + \text{finite part}. \tag{26}$$

The last term in equation (23), namely the cross-terms, contain finite contributions as well as "nonlocal" divergences.

$$\Gamma_{\text{nonlocal}} = \Gamma^3_{\text{fin}} + \Gamma^{\text{div}}_{\text{nonlocal}}. \tag{27}$$

These nonlocal divergence contributions will always appear due to the divergent subdiagrams (beyond two-point functions) contained in the graph. As we will show next in a particular example, the renormalization of previous orders will always allow one to cancel these contributions if the theory is renormalizable. In the present scheme the result is automatic and follows from the operation we have

just defined, in an algebraic manner. There is no need for graphic representations of relevant contributions, although it is possible.

The renormalized amplitude say, in *nth* loop order, can therefore be defined as

$$\begin{aligned}\Gamma_{\mathrm{R}}^{(n)} &= T^{D}\Gamma^{(n)} - \Gamma_{\mathrm{local}}^{\mathrm{div}(n)} - \Gamma_{\mathrm{nonlocal}}^{\mathrm{div}(n)} \\ &= \Gamma_{\mathrm{fin}}^{1(n)} + \Gamma_{\mathrm{fin}}^{2(n)} + \Gamma_{\mathrm{fin}}^{3(n)}\end{aligned} \tag{28}$$

where the contributions $\Gamma_{\mathrm{local}}^{\mathrm{div}(n)}$ and $\Gamma_{\mathrm{nonlocal}}^{\mathrm{div}(n)}$ contain the counterterms typical of order n as well as the counterterms coming from divergent subdiagrams of previous order as will become clear in the examples. Notice from the equation above that our framework automatically delivers the counterterms

$$\Gamma_{\mathrm{CT}}^{1} = -\Gamma_{\mathrm{local}}^{\mathrm{div}} - \Gamma_{\mathrm{nonlocal}}^{\mathrm{div}} \tag{29}$$

and just as in BPHZ, subtracting off the necessary counterterms leaves us with the finite part of the amplitude. The main difference between our method and BPHZ is that we never modify the original Feynman amplitude, since we use an identity at the level of the integrand and BPHZ, a Taylor expansion. An immediate consequence of this difference is that in the present procedure symmetries can always be preserved as has been shown in Battistel, 1999; Battistel *et al.*, 1998; Brizda *et al.*, 1999; Sampaio *et al.*, 2002; Scarpelli *et al.*, 2001.

Now we proceed to evaluate the second class of diagrams, namely those which contain two-point functions as subdiagrams. Let us call U all the two-point diagrams contained in a given amplitude Γ. It is easy to see that they can be factored out inside of the total amplitude in the following sense

$$\Gamma = \sum_{\mathrm{all}} \prod_{\sum_j \in U} \mathcal{R}_j \Sigma_j^{(l)}\left(q_j^2\right) \tag{30}$$

where $\mathcal{R}_j$ stands for the remaining pieces in the amplitude, j characterizes a specific two-point function, q_j is one of the integration momenta (but external to $\sum_j$). Now since the operation $T^D\Gamma$ is an identity, i.e. $T^D\Gamma = \Gamma$ we can define the partially renormalized amplitude (with all two-point function subdiagrams properly renormalized) as follows

$$\bar{\Gamma} = \Gamma + \Gamma_{\mathrm{CT}}^{2} \tag{31}$$

therefore we have

$$\Gamma_{\mathrm{CT}}^{2} = \sum_{\mathrm{all}} \prod_{\sum_j \in U} \mathcal{R}_j \left[\delta_j^{(l)} m^2 - A_j^{(l)} q_j^2\right] \tag{32}$$

and Γ_{CT}^{2} are all counterterms characteristic subdiagrams involving two-point functions. $\delta_j^{(l)} m^2$ stands for the mass renormalization and $A_j^{(l)}$ for the wave function renormalization. Explicit expressions for these objects will be given in the following section where a specific example is worked out. In order to get the renormalized

amplitude of order n from $\bar{\Gamma}$ one proceeds in the same way as for diagrams of class one defined above. We thus have

$$\begin{aligned}\Gamma_R &= T^D\bar{\Gamma} - \bar{\Gamma}^{\text{div}}_{\text{local}} - \bar{\Gamma}^{\text{div}}_{\text{nonlocal}} \\ &= \bar{\Gamma}^1_{\text{fin}} + \bar{\Gamma}^2_{\text{fin}} + \bar{\Gamma}^3_{\text{fin}}.\end{aligned} \tag{33}$$

The whole procedure will become apparent in the concrete example of the following section.

3. $\lambda\phi_6^3$ THEORY AS AN EXAMPLE

Consider the $\lambda\phi_6^3$ theory Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi_0(x))^2 - m_0^2\phi_0^2(x)\right] - \frac{\lambda_0}{3!}\phi_0^3(x) \tag{34}$$

It is easy to show that a Feynman graph in this theory has the superficial degree of divergence D written as

$$D = 6 - 2N \tag{35}$$

where N is the number of external legs. This means that only Green's functions with $N \leq 3$ are divergent. For the one-point functions we will assume that we can impose the condition $\langle 0 \mid \hat{\phi} \mid 0\rangle = 0$ at all orders and we will not worry about one-point diagrams. We will just work with the two- and three-point Green's functions which possess quadratic and logarithmic divergences.

We will effect the renormalization through the redefinition of the Lagrangian parameters as:

$$\phi_0 = \sqrt{Z_\phi\phi} \tag{36}$$

$$m_0^2 = Z_m m^2 \tag{37}$$

$$\lambda_0 = Z_\lambda\lambda \tag{38}$$

which allow the Lagrangian to be rewritten as

$$\mathcal{L} = \mathcal{L}_{\text{F}} + \mathcal{L}_{\text{CT}} \tag{39}$$

where

$$\mathcal{L}_{\text{F}} = \frac{1}{2}\left[(\partial_\mu\phi^2 - m^2\phi^2\right] - \frac{\lambda}{3!}\phi^3 \tag{40}$$

and

$$\mathcal{L}_{\text{CT}} = \frac{1}{2}\left[(Z_\phi - 1)(\partial_\mu\phi)^2 - (Z_\phi Z_m - 1)m^2\phi^2\right] - \left(Z_\phi^{3/2}Z_\lambda - 1\right)\frac{\lambda}{3!}\phi^3 \tag{41}$$

At the *nth* order one has

$$\mathcal{L}_{\rm CT} = \mathcal{L}_{\rm CT}^{(1)} + \mathcal{L}_{\rm CT}^{(2)} + \ldots \mathcal{L}_{\rm CT}^{(n)} \tag{42}$$

We effect the renormalization at each order imposing the conditions:

- Relative to the propagator

$$D_{\rm R}^{-1}(0) = -m^2 \tag{43}$$

and

$$\left| \frac{\partial}{\partial p^2} D_{\rm R}^{-1}(p^2) \right|_{p=0} = 1 \tag{44}$$

- Relative to vertex function

$$-i\,M_{\rm R}(0) = -i\lambda(1 + \text{finite corrections}) \tag{45}$$

We can rewrite the bare Lagrangian (34) as

$$\mathcal{L} = \frac{1}{2}\left[(1+A)(\partial_\mu\phi)^2 - (m^2 + \delta m^2)\phi^2\right] - (1+B)\frac{\lambda}{3!}\phi^3. \tag{46}$$

in order to identify the renormalization constants

$$Z_\phi^{(n)} = 1 + A^{(n)} \tag{47}$$

$$Z_m^{(n)} = \frac{1}{Z_\phi^{(n)} m^2}\left(m^2 + \delta^{(n)} m^2\right) \tag{48}$$

$$Z_\lambda^{(n)} = \frac{1 + B^{(n)}}{\left(Z_\phi^{(n)}\right)^{3/2}} \tag{49}$$

at each order by imposing renormalization conditions. Since, in practice we renormalize each diagram of the given order, the counterterms can be written as

$$B^{(n)} = \sum_{j=1}^{a} + B_j^{(n)} \tag{50}$$

$$A^{(n)} = \sum_{j=1}^{b} A_j^{(n)} \tag{51}$$

$$\delta^{(n)} m^2 = \sum_{j=1}^{b} \delta_j^{(n)} m^2 \tag{52}$$

here a, (b) is the number of three (two)-point diagrams which contribute to order n.

At the *nth* order the inverse propagator function is written as

$$D_{\rm R}^{-1}(p^2) = p^2 - m^2 - \Sigma_{\rm R}^{(1)}(p^2) - \Sigma_{\rm R}^{(2)}(p^2) \ldots$$

$$-\delta^{(n)}m^2 + A^{(n)}p^2 - \Sigma^{(n)}(p^2) \tag{53}$$

and the vertex function as

$$\begin{aligned} -iM_{\mathrm{R}}(p, p') = -i\lambda\{1 + V_{\mathrm{R}}^{(1)}(p, p') + V_{\mathrm{R}}^{(2)}(p, p') \dots \\ + V^{(n)}(p, p') + B^{(n)}\}. \end{aligned} \tag{54}$$

Using the technique in each diagram contained in the $\Sigma^{(n)}(p^2)$ and in the $V^{(n)}(p, p')$ amplitudes we separate the local divergent part and identify all divergent substructures. Imposing renormalization conditions we can always identify $A^{(n)}$, $\delta^{(n)}m^2$, and $B^{(n)}$.

In order to identify the counterterms of the order in question and to write the nonlocal ones in terms of divergences of lower orders, showing thus that one need not worry about them, it is convenient to define the following functions:

- Relative to vertex correction counterterms (type j diagrams)

$$\begin{aligned} iB_j^{(n)} &= (-i\lambda)^{2n+1}(i)^{3n} I_{\log 1}^{(n)}(m^2, \Lambda) \\ &= \Gamma_{\text{local}}^{\text{div}(n)} \end{aligned} \tag{55}$$

  where

$$I_{\log 1}^{(n)}(m^2, \Lambda) = \prod_{i=1}^{n} \int_{\Lambda} \frac{d^6 k_i}{(2\pi)^6} \Upsilon^{(n)}(k_1, k_2, \dots k_n, m^2) \tag{56}$$

  with

$$\begin{aligned} &\Upsilon^{(n)}(k_1, k_2, \dots k_n, m^2) \\ &= \frac{1}{\left(k_1^2 - m^2\right)^3} \left(\prod_{j=2}^{n} \frac{1}{\left(k_j^2 - m^2\right)^2} \right) Q(k_i, k_{i+1}, m^2). \end{aligned} \tag{57}$$

  For $n = 1$

$$Q = 1, \tag{58}$$

  otherwise

$$Q = \prod_{i=1}^{n-1} \left\{ \frac{1}{[(k_i - k_{i+1})^2 - m^2]} \right\}. \tag{59}$$

  Notice that what we have defined here are generalizations of the simple one-loop logarithmically divergent integral $I_{\log}(m^2)$ which we encountered in our one-loop example.

- Relative to all finite contributions to vertex corrections (type j diagram), which correspond to the renormalized diagram

$$\Gamma_{\mathrm{R}}^{(n)} = \Gamma_{\text{fin}}^{1(n)} - \Gamma_{\text{fin}}^{3(n)}$$

$$= (-i\lambda)^{2n+1}(i)^{3n}\prod_{i=1}^{n}\int\frac{d^6k_i}{(2\pi)^6}\,\Xi^{(n)}(k_1,\ldots,k_n,\,p,\,p',\,m^2). \quad (60)$$

- Relative to the finite contribution, defined in Eq. (24) for the overlapping diagrams

$$\Gamma^{1(n)}_{\text{fin}} = (\Pi R)\prod_{i=1}^{n} f_i^{\text{fin}}(p, k_i, m^2)$$

$$= \frac{(-i\lambda)^{2n}(i)^{3n-1}}{2}\prod_{i=1}^{n}\int\frac{d^6k_i}{(2\pi)^6}\Theta^{(n)}(k_1,\ldots,k_n,\,p,\,m^2). \quad (61)$$

In each order there will appear new types of divergent integrals. Therefore throughout the text we will define some new divergent integrals similar to the ones above (Eq. (56)). These quantities are always independent of external momenta. Next we apply the procedure to all diagrams up to two loops in order to exemplify how the method works.

To *nth* order it suffices to treat four cases, the first related to the vertex function and the others to the self-energy, which contain the overlapping divergences, two-point functions as subdivergences, and nested two-point functions.

3.1. Three-Point Functions

3.1.1. The One-Loop Order

The vertex correction has only one contribution at one loop level whose diagram is depicted in Fig. 1. The corresponding amplitude is

$$\Gamma = -iV^{(1)}(p,\,p')$$

$$= \lambda^3\int_\Lambda\frac{d^6k}{(2\pi)^6}\,\frac{1}{(k^2-m^2)[(p-k)^2-m^2][(p'-k)^2-m^2]}. \quad (62)$$

Using the notation introduced in section 2 we write

$$-iV^{(1)}(p,\,p') = \int_\Lambda\frac{d^6k}{(2\pi)^6}R(k,m^2,\lambda)\,f(p,k,m^2)\,f(p',k,m^2) \quad (63)$$

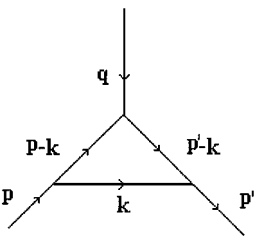

Fig. 1. One-loop vertex correction $-iV^{(1)}(p,\,p')$.

with

$$R(k, m^2, \lambda) = \frac{\lambda^3}{(k^2 - m^2)}. \tag{64}$$

According to IRT we write, given that the divergence is logarithmic and therefore $D = 0$

$$-i T^0 V^{(1)}(p, p') = \Gamma^{\text{div}}_{\text{local}} + \Gamma^1_{\text{fin}} + \Gamma^3_{\text{fin}} \tag{65}$$

(recall that in this case $\Gamma^2_{\text{fin}} = \Gamma^{\text{div}}_{\text{nonlocal}} = 0$) where

$$\begin{aligned}
\Gamma^{\text{div}}_{\text{local}} &= \int_\Lambda \frac{d^6k}{(2\pi)^6} R(k, m^2, \lambda) f^0(k, m^2) f^0(k, m^2) \\
&= \lambda^3 \int_\Lambda \frac{d^6k}{(2\pi)^6} \frac{1}{(k^2 - m^2)^3} \\
&= \lambda^3 I^{(1)}_{\log 1}(m^2, \Lambda) \\
&= i B^{(1)}
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
\Gamma_{\text{R}} &= \Gamma^1_{\text{fin}} + \Gamma^3_{\text{fin}} \\
&= \lambda^3 \int \frac{d^6k}{(2\pi)^6} \Xi^{(1)}(k, p, p', m^2)
\end{aligned} \tag{67}$$

with

$$\begin{aligned}
\lambda^3 \Xi^{(1)} R(k, p, p', m^2) = R(k, m^2, \lambda)\{ & f^{\text{fin}}(k, p, m^2) f^{\text{fin}}(k, p', m^2) \\
& + f^0(k, m^2) f^{\text{fin}}(k, p', m^2) \\
& + f^{\text{fin}}(k, p, m^2) f^0(k, m^2)\}.
\end{aligned} \tag{68}$$

Notice that the finite part of this diagram contains the cross-terms $f_0 . f_{\text{fin}}$ in since its integral is finite.

3.1.2. The Two-Loop Order

Three diagram types contribute to the vertex correction at two loops. The total amplitude can be written as

$$-i V^{(2)}(p, p') = -3i V^{(2)}_1(p, p') - i V^{(2)}_2(p, p') - i V^{(2)}_3(p, p'). \tag{69}$$

In this order the counterterms will be identified as

$$B^{(n)} = 3B^{(2)}_1 + B^{(2)}_2 + B^{(2)}_3. \tag{70}$$

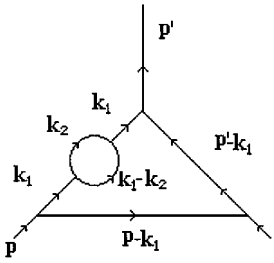

Fig. 2. The two-loop vertex correction contribution $-iV_1^{(2)}(p, p')$.

The first amplitude $iV_1^{(2)}(p, p')$ corresponds to the diagram in Fig. 2. This diagram contains a quadratic divergent subdiagram (a first-order, two-point function correction). It can be completely separated in terms of the internal momentum k_1 as mentioned before. Then

$$\begin{aligned}\Gamma &= -iV_1^{(2)}(p, p') \\ &= \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \frac{(-i\lambda)^3(i)^4}{\left(k_1^2 - m^2\right)^2[(p-k_1)^2 - m^2][(p'-k_1)^2 - m^2]} \\ &\quad \times \left\{i\Sigma^{(1)}\left(k_1^2\right)\right\} \end{aligned} \tag{71}$$

where $i\sum^{(1)}(k_1^2)$ is the one-loop self-energy amplitude. The one-loop renormalized self-energy is

$$\Sigma_{\mathrm{R}}^{(1)}\left(k_1^2\right) = \Sigma_{\mathrm{CT}}^{(1)}\left(k_1^2\right) + \Sigma^{(1)}\left(k_1^2\right) \tag{72}$$

where

$$\Sigma_{\mathrm{CT}}^{(1)}\left(k_1^2\right) = \delta^{(1)}m^2 - A^{(1)}k_1^2. \tag{73}$$

Thus the amplitude containing no two-point function substructure is directly obtained as

$$\begin{aligned}\bar{\Gamma} &= \Gamma + \Gamma_{\mathrm{CT}}^2 \\ &= \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \frac{(-i\lambda)^3(i)^4}{\left(k_1^2 - m^2\right)^2[(p-k_1)^2 - m^2][(p'-k_1)^2 - m^2]} \\ &\quad \times \left\{i\Sigma_{\mathrm{R}}^{(1)}\left(k_1^2\right)\right\}. \end{aligned} \tag{74}$$

All possible nonlocal divergences in this case will be canceled when we consider the one-loop renormalization. Next we use the IRT for the logarithmic divergence. In our notation we obtain

$$-iT^0\bar{\Gamma} = \bar{\Gamma}_{\mathrm{local}}^{\mathrm{div}} + \bar{\Gamma}_{\mathrm{fin}}^1 + \Gamma_{\mathrm{fin}}^3 \tag{75}$$

with

$$\int \frac{d^6k_2}{(2\pi)^6} R(k_1, k_2, m^2, \lambda) = \frac{(-i\lambda)^3 (i)^4}{\left(k_1^2 - m^2\right)^2} \left\{ i \Sigma_R^{(1)} \left(k_1^2\right) \right\} = R(k_1, m^2, \lambda). \quad (76)$$

The explicit expression for $i \sum_R^{(1)}(k_1^2)$ will be given in Eq. (107). Using the above expression for $R(k_1, m^2, \lambda)$

$$\begin{aligned} \bar{\Gamma}^{\text{div}}_{\text{local}} &= \int_\Lambda \frac{d^6k_2}{(2\pi)^6} R(k_1, m^2, \lambda) f^0(k_1, m^2) f^0(k_1, m^2) \\ &= (-i\lambda)^3 (i)^4 I^{(2)}_{\log\, 2}(m^2, \lambda^2, \Lambda) = i B_1^{(2)}. \end{aligned} \quad (77)$$

We have just defined another logarithmic divergent quantity which is characteristic of the two-loop order. Note the explicit appearance of coupling constant. This should emphasize the fact that the amplitude depends on a two-point function subdiagram, which has been properly renormalized. All counterterms possessing such type of subdiagram will look like this.

$$I^{(2)}_{\log\, 2}(m^2, \lambda^2, \Lambda) = \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \frac{1}{\left(k_1^2 - m^2\right)^4} \left\{ i \Sigma^{(1)}_{\text{R}} \left(k_1^2\right) \right\} \quad (78)$$

The finite part is

$$\begin{aligned} \bar{\Gamma}^1_{\text{fin}} + \bar{\Gamma}^3_{\text{fin}} = \int \frac{d^6k_1}{(2\pi)^6} R(k_1, m^2, \lambda) \{ & f^0(k_1, m^2) f^{\text{fin}}(p', k_1, m^2) \\ & + f^{\text{fin}}(p, k_1, m^2) f^0(k_1, m^2) \\ & + f^{\text{fin}}(p, k_1, m^2) f^{\text{fin}}(p', k_1, m^2) \}. \end{aligned} \quad (79)$$

It is not necessary to give explicit expressions for the finite part and therefore we make explicit the divergent contributions only.

Now we consider the diagram corresponding to the second contribution $-i V_2^{(2)}(p, p')$ which belongs to class one (Fig. 3). The amplitude reads

$$\begin{aligned} -i V_2^{(2)}(p, p') = \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} R(k_1, k_2, m^2, \lambda) \\ \times f(p, k_1, m^2) f(p', k_1, m^2) f(p', k_2, m^2) \end{aligned} \quad (80)$$

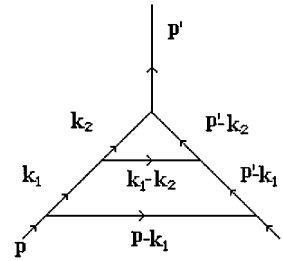

Fig. 3. The two-loop vertex correction contribution $-i V_2^{(2)}(p, p')$.

with

$$R(k_1, k_2, m^2, \lambda) = \frac{(-i\lambda)^5 (i)^6}{\left(k_1^2 - m^2\right)\left(k_2^2 - m^2\right)[(k_1 - k_2)^2 - m^2]}. \tag{81}$$

Using the IRT we have

$$-iT^0 V_1^{(2)}(p, p') = \Gamma^{\text{div}}_{\text{local}} + \Gamma^1_{\text{fin}} + \Gamma_{\text{nonlocal}} \tag{82}$$

where

$$\begin{aligned}\Gamma^{\text{div}}_{\text{local}} &= (-i\lambda)^5 (i)^6 I^{(2)}_{\log\,1}(m^2, \Lambda) = iB_2^{(2)} \\ &= \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} \frac{(-i\lambda)^5 (i)^6}{\left(k_1^2 - m^2\right)^3 \left(k_2^2 - m^2\right)^2 [(k_1 - k_2)^2 - m^2]} \\ &= (-i\lambda)^5 (i)^6 \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} \Upsilon^{(2)}(k_1, k_2, m^2). \end{aligned} \tag{83}$$

In this type of structure (to all orders) the nonlocal contribution Γ_{nonlocal} will have the form

$$\Gamma_{\text{nonlocal}} = \Gamma^3_{\text{fin}} + \Gamma^{\text{div}}_{\text{nonlocal}} \tag{84}$$

and in this case we have

$$\Gamma^{\text{div}}_{\text{nonlocal}} = (-i\lambda)^5 (i)^6 \int = \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} \Xi^{(1)}(k_1, p, p', m^2) \Upsilon^{(1)}(k_2, m^2). \tag{85}$$

Note that this term is completely written in terms of one-loop contributions already considered. Therefore it poses no problem to renormalization. This particular example illustrates a basic difference between the present method and others: The subdivergences need not be previously identified. They appear algebraically. In cases where it is simple to identify the subdivergences, this is not necessarily a great advantage. However in higher orders it might become considerably simpler to identify all divergent substructures in an algebraic fashion. In fact, as will become clear in what follows, the procedure is designed to display all relevant (to renormalization) subdivergences. The finite contributions can be written as

$$\begin{aligned}\Gamma^{(2)}_{\text{R}} &= \Gamma^{(1)}_{\text{fin}} + \Gamma^{(3)}_{\text{fin}} \\ &= (-i\lambda)^5 (i)^6 \int \frac{d^6k_1}{(2\pi)^6} \int \frac{d^6k_2}{(2\pi)^6} \Xi^{(1)}(k_1, k_2, p, p', m^2). \end{aligned} \tag{86}$$

with

$$\begin{aligned}\Xi &= (-i\lambda)^5 (i)^6 \Xi^{(2)}(k_1, k_2, p, p', m^2) \\ \Xi &= R(k_1, k_2, m^2, \lambda) \times \{ f^{\text{fin}}(k_1, p, m^2) f^{\text{fin}}(k_1, p', m^2) f^{\text{fin}}(k_2, p', m^2)\end{aligned}$$

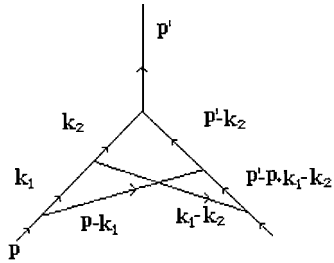

Fig. 4. The two-loop vertex correction contribution $-iV_3^{(2)}(p, p')$.

$$
\begin{aligned}
&+ f^0(k_1, m^2) f^0(k_1, m^2) f^{\text{fin}}(k_2, p', m^2) \\
&+ f^0(k_1, m^2) f^{\text{fin}}(k_1, p', m^2) f^{\text{fin}}(k_2, p', m^2) \\
&+ f^{\text{fin}}(k_1, p, m^2) f^0(k_1, m^2) f^{\text{fin}}(k_2, p', m^2)\} \\
&+ \frac{(-i\lambda)^5 (i)^6 \left(2k_1 \cdot k_2 - k_1^2\right) \Xi^{(1)}(k_1, p, p', m^2)}{\left(k_2^2 - m^2\right)^3 [(k_1 - k_2)^2 - m^2]}. \qquad (87)
\end{aligned}
$$

The last term in the above equation is obtained by using the operation (12) considering k_1 as external momentum. This is necessary to identify the one-loop structure.

The last two-loop diagram $-iV_3^{(2)}(p, p')$ is depicted in Fig. 4. The corresponding amplitude is

$$
\begin{aligned}
-iV_3^{(2)}(p, p') = \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} \{ & R(k_1, k_2, m^2, \lambda) f(p, k_1, m^2) \\
& \times f(p', k_2, m^2) f(p - p', k_1 - k_2, m^2)\} \qquad (88)
\end{aligned}
$$

with

$$
R(k_1, k_2, m^2, \lambda) = \frac{(-i\lambda)^5 (i)^6}{\left(k_1^2 - m^2\right)\left(k_2^2 - m^2\right)[(k_1 - k_2)^2 - m^2]}. \qquad (89)
$$

Using the IRT we have

$$
-iT^0 V_1^{(2)}(p, p') = \Gamma_{\text{local}}^{\text{div}} + \Gamma_{\text{R}} \qquad (90)
$$

where

$$
\begin{aligned}
\Gamma_{\text{local}}^{\text{div}} &= i\lambda^5 \int_\Lambda \frac{d^6k_1}{(2\pi)^6} \int_\Lambda \frac{d^6k_2}{(2\pi)^6} \frac{1}{\left(k_1^2 - m^2\right)^2 \left(k_2^2 - m^2\right)^2 [(k_1 - k_2)^2 - m^2]^2} \\
&= i\lambda^5 I_{\log\,3}^{(2)}(m^2, \Lambda) = iB_3^{(2)}. \qquad (91)
\end{aligned}
$$

We defined above another logarithmic divergent quantity. This diagram type is often called a primitively divergent diagram. Note that there are no subdivergences.

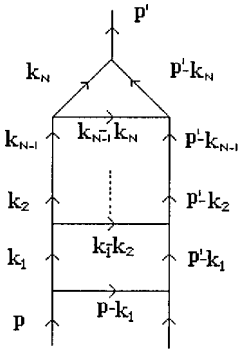

Fig. 5. The n-loop vertex correction contribution $-iV_1^{(n)}(p, p')$.

3.1.3. The N-Loop Order

As discussed before we now consider only one contribution of each kind. The vertex type contribution depicted in Fig. 5 is the first one. It will appear as a substructure of the overlapping self-energy diagram which we will also consider.

The amplitude corresponding to the vertex correction in Fig. 5 is

$$-iV_1^{(n)}(p, p') = (\Pi R)(\Pi f) \tag{92}$$

with

$$(\Pi R) = (-i\lambda)^{2n+1}(i)^{3n}\left\{\prod_{j=1}^{n}\int_{\Lambda}\frac{d^6k_j}{(2\pi)^6}\frac{1}{k_j^2 - m^2}\right\}Q(k_1, k_{i+1}, m^2) \tag{93}$$

where Q is the same function as defined in (58) and (59). The subscript 1 in $V_1^{(n)}(p, p')$ refers to the fact that only one diagram is being considered (type 1). The external momentum dependent part (Πf) is given by

$$(\Pi f) = \left\{\frac{1}{(p - k_1)^2 - m^2}\right\}\prod_{j=1}^{n}\left\{\frac{1}{(p' - k_j)^2 - m^2}\right\}. \tag{94}$$

Using the IRT, we get

$$T^0(\Pi f) = \{f^0(k_1, m^2) + f^{\text{fin}}(k_1, p, m^2)\} \times \prod_{j=1}^{n}\{f^0(k_j, m^2) + f^{\text{fin}}(k_j, p', m^2)\}. \tag{95}$$

In the same way we have

$$-iT^0V_1^{(n)}(p, p') = \Gamma_{\text{local}} + \Gamma_{\text{fin}}^1 + \Gamma_{\text{nonlocal}}. \tag{96}$$

Since $\Gamma^2_{\text{fin}} = 0$ we can write

$$\begin{aligned}\Gamma^{\text{div}}_{\text{local}} &= iB^{(n)}_1 \\ &= (\Pi R)^0_1(k_1, m^2)\prod_{j=1}^{n} f^0_j(k_j, m^2) \\ &= (-i\lambda)^{2n+1}(i)^{3n} I^{(n)}_{\log\,1}(m^2, \Lambda) \\ &= (-i\lambda)^{2n+1}(i)^{3n}\prod_{j=1}^{n}\frac{d^6k_j}{(2\pi)^6}\Upsilon^n(k_1, k_2, \ldots k_n, m^2) \end{aligned} \tag{97}$$

and

$$\Gamma_{\text{nonlocal}} = \Gamma^3_{\text{fin}} + \Gamma^{\text{div}}_{\text{nonlocal}} \tag{98}$$

where

$$\begin{aligned}\Gamma^{\text{div}}_{\text{nonlocal}} &= (-i\lambda)^{2n+1}(i)^{3n}\prod_{j=1}^{n}\frac{d^6k_j}{(2\pi)^6} \\ &\quad\times\sum_{a=1}^{n-1}\Xi^{(a)}(k_1, \ldots, k_a; p, p', m^2)\Upsilon^{(n-a)}(k_{a+1}, \ldots, k_n; m^2). \end{aligned} \tag{99}$$

Note that the above equation contains subdivergences which have already appeared at lower orders and have already been included in the Lagrangian. Here we clearly see that the application of the method displays all the subdivergencies in an algebraic way. Moreover it stresses the inductive character of the method. If we assume that the theory is renormalized at $(n-1)$th order, the contribution at *nth* order will solely depend on structures (finite and divergent) which have already played their role at lower orders. Also it is noteworthy that *all* divergencies and finite parts of *all* previous orders play an important role at *nth* order. The above expression clearly displays a difference between this method and BPHZ. In BPHZ one would have to deal with the detailed topology of the diagram first. Here the counterterms appear algebraically. This simplicity may be due to the example we are working with. However, it constitutes even in this case a notorious simplification.

3.2. Two-Point Functions

3.2.1. The One-Loop Order

The self-energy has only one diagram contribution at one-loop level which we identify in Fig. 6. It corresponds to the amplitude

$$i\Sigma^{(1)}(p^2) = \frac{\lambda^2}{2}\int_\Lambda \frac{d^6k_j}{(2\pi)^6}\frac{1}{(k^2-m^2)[(p-k)^2-m^2]} \tag{100}$$

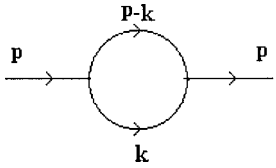

Fig. 6. One-loop self-energy $i\,\Sigma^{(1)}(p^2)$.

where

$$R(k, m^2, \lambda) = \frac{\lambda^2}{2}\frac{1}{(k^2 - m^2)}. \tag{101}$$

Using IRT we have

$$(T^2) i\,\Sigma^{(1)}(p^2) = \Gamma^{\text{div}}_{\text{local}} + \Gamma^{1}_{\text{fin}} \tag{102}$$

with

$$\Gamma^{\text{div}}_{\text{local}} = \frac{\lambda^2}{2}\left\{ I^{(1)}_{\text{quad}}(m^2, \Lambda) + p^2\left[g_{\mu\nu}\frac{4}{6} I^{\mu\nu(1)}_{\text{log}}(m^2, \Lambda) - I^{(1)}_{\text{log }1}(m^2, \Lambda)\right]\right\}, \tag{103}$$

where we have defined

$$I^{(1)}_{\text{quad}}(m^2, \Lambda) = \int_{\Lambda} \frac{d^6k}{(2\pi)^6}\frac{1}{(k^2 - m^2)^2} \tag{104}$$

and

$$I^{\mu\nu(1)}_{\text{log}}(m^2, \Lambda) = \int_{\Lambda} \frac{d^6k}{(2\pi)^6}\frac{k^\mu k^\nu}{(k^2 - m^2)^4}. \tag{105}$$

The finite part is

$$\begin{aligned}\Gamma^{1}_{\text{fin}} = \Gamma_{\text{R}} &= \frac{\lambda^2}{2}\int \frac{d^6k}{(2\pi)^6}\Theta^{(1)}(k, p, m^2) \\ &= \frac{\lambda^2}{2}\left\{ \int \frac{d^6k}{(2\pi)^6}\frac{(p)^4}{(k^2 - m^2)4} \right. \\ &\quad \left. + \int \frac{d^6k}{(2\pi)^6}\frac{(2p\cdot k - p^2)^3}{(k^2 - m^2)^4[(p-k)^2 - m^2]}\right\}. \end{aligned} \tag{106}$$

The explicit calculation of the integral in the above equation leads to

$$\Gamma_{\text{R}} = i\,\Sigma^{(1)}_{\text{R}}(p^2) = \frac{\lambda^2}{4}\frac{i}{(4\pi)^3}\left\{ (p^2 - 3m^2)F(m^2, p^2) - \frac{p^2}{2}\right\} \tag{107}$$

where $F(m^2, p^2)$, for $p^2 < 4m^2$, is given by

$$F(m^2, p^2) = \frac{\sqrt{4m^2 - p^2}}{|p|}\left[2\arctan\left(\frac{\sqrt{4m^2 - p^2}}{|p|}\right) + \pi\right] - 2 \tag{108}$$

and for $p^2 > 4m^2$,

$$F(m^2, p^2) = -\frac{\sqrt{p^2 - 4m^2}}{|p|}\left[\ln\left(\frac{|p| - \sqrt{p^2 - 4m^2}}{|p| + \sqrt{p^2 - 4m^2}}\right) + i\pi\right] - 2. \tag{109}$$

We now summarize the results obtained so far for one-loop renormalization,

$$A^{(1)} = \frac{i\lambda^2}{2}\left\{I^{(1)}_{\log 1}(m^2, \Lambda) - \frac{4}{6} g_{\mu\nu} I^{\mu\nu(1)}_{\log}(m^2, \Lambda)\right\} \tag{110}$$

$$\delta^{(1)} m^2 = \frac{\lambda^2}{2} i I^{(1)}_{\text{quad}}(m^2, \Lambda) \tag{111}$$

and

$$B^{(1)} = -i\lambda^2 I^{(1)}_{\log 1}(m^2, \Lambda) \tag{112}$$

where $I^{(1)}_{\log\ 1}(m^2, \Lambda)$, $I^{\mu\nu(1)}_{\log}(m^2, \Lambda)$, and $I^{(1)}_{\text{quad}}(m^2, \Lambda)$ are defined in (56), (105), and (104), respectively.

3.2.2. The Two-Loop Order

Two types of diagram contribute to the self-energy correction at two loops. The total amplitude can be written as

$$\Sigma^{(2)}(p^2) = 2\Sigma^{(2)}_1(p^2) + \Sigma^{(2)}_2(p^2). \tag{113}$$

Here the counterterms to be identified are

$$A^{(2)} = 2A^{(2)}_1 + A^{(2)}_2 \tag{114}$$

$$\delta^{(2)} m^2 = 2\delta^{(2)}_1 m^2 + \delta^{(2)}_2 m^2. \tag{115}$$

The first amplitude $i\,\Sigma^{(2)}_1(p^2)$ corresponds to the diagram in Fig. 7. This is the same case that we have seen in Eq. (71). Considering the one-loop renormalization we can write

$$\bar{\Gamma} = i\,\bar{\Sigma}^{(2)}_1(p^2) = \int \frac{d^6k}{(2\pi)^6} \frac{(-i\lambda)^2 (i)^3}{\left(k_1^2 - m^2\right)^2 [(p - k_1)^2 - m^2]} \left\{i\,\Sigma^{(1)}_{\text{R}}\left(k_1^2\right)\right\}. \tag{116}$$

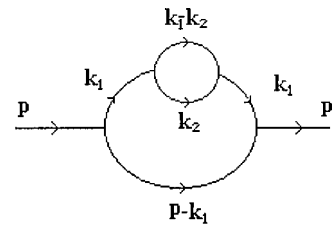

Fig. 7. The two-loop self-energy contribution $i\,\Sigma^{(2)}_1(p^2)$.

Then we apply IRT and obtain

$$i T^2 \bar{\Sigma}_1^{(2)}(p^2) = \bar{\Gamma}^{\text{div}}_{\text{local}} + \bar{\Gamma}^1_{\text{fin}} \tag{117}$$

in which

$$\bar{\Gamma}^{\text{div}}_{\text{local}} = \lambda^2 i \left\{ I^{(2)}_{\text{quad }2}(m^2, \lambda^2, \Lambda) + p^2 \left[\frac{4 g_{\mu\nu}}{6} I^{\mu\nu(2)}_{\text{log2}}(m^2, \lambda^2, \Lambda) - I^{(2)}_{\text{log2}}(m^2, \lambda^2, \Lambda) \right] \right\} \tag{118}$$

and

$$I^{(2)}_{\text{quad2}}(m^2, \lambda^2, \Lambda) = \int_\Lambda \frac{d^6 k_1}{(2\pi)^6} \frac{1}{\left(k_1^2 - m^2\right)^3} \left\{ i \Sigma^{(1)}_{\text{R}}\left(k_1^2\right) \right\} \tag{119}$$

$$I^{\mu\nu(2)}_{\text{log2}}(m^2, \lambda^2, \Lambda) = \int_\Lambda \frac{d^6 k_1}{(2\pi)^6} \frac{k^\mu k^\nu}{\left(k_1^2 - m^2\right)^5} \left\{ i \Sigma^{(1)}_{\text{R}}\left(k_1^2\right) \right\} \tag{120}$$

whereas

$$\bar{\Gamma}^1_{\text{fin}} = \lambda^2 i \int \frac{d^6 k_1}{(2\pi)^6} \left\{ \frac{p^4}{\left(k_1^2 - m^2\right)^5} + \frac{(2p \cdot k - p^2)^3}{\left(k_1^2 - m^2\right)^4 [(p - k_1)^2 - m^2]} \right\} \left\{ i \Sigma^{(1)}_{\text{R}}\left(k_1^2\right) \right\}. \tag{121}$$

The second amplitude $i \Sigma^{(2)}_{(2)}(p^2)$ corresponds to the diagram in Fig. 8. It reads

$$i \Sigma^{(2)}_{(2)}(p^2) = \int_\Lambda \frac{d^6 k_1}{(2\pi)^6} \int_\Lambda \frac{d^6 k_2}{(2\pi)^6} R(k_1, k_2, m^2, \lambda) f(p, k_1, m^2) f(p, k_2, m^2) \tag{122}$$

with

$$R(k_1, k_2, m^2, \lambda) = \frac{1}{2} \left\{ \frac{(-i\lambda)^4 (i)^5}{\left(k_1^2 - m^2\right)\left(k_2^2 - m^2\right)[(k_1 - k_2)^2 m^2]} \right\}. \tag{123}$$

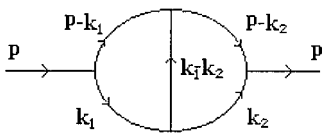

Fig. 8. The two-loop self-energy contribution $i \Sigma^{(2)}_2(p^2)$.

The same procedure enables us to write

$$\Gamma^{\rm div}_{\rm local} = \frac{i\lambda^4}{2}\left\{ I^{(2)}_{\rm quad\ 1}(m^2,\Lambda) + 2p^2\left[\frac{4g_{\mu\nu}}{6} I^{\mu\nu(2)}_{\rm log1}(m^2,\Lambda) - I^{(2)}_{\rm log1}(m^2,\Lambda)\right]\right\} \tag{124}$$

where

$$I^{(2)}_{\rm quad\ 1}(m^2,\Lambda) = \int_\Lambda \frac{d^6k_1}{(2\pi)^6}\int_\Lambda \frac{d^6k_2}{(2\pi)^6} \times \frac{1}{\left(k_1^2-m^2\right)^2\left(k_2^2-m^2\right)^2[(k_1-k_2)^2-m^2]} \tag{125}$$

$$I^{\mu\nu(2)}_{\rm log1}(m^2,\Lambda) = \int_\Lambda \frac{d^6k_1}{(2\pi)^6}\int_\Lambda \frac{d^6k_2}{(2\pi)^6} \times \frac{k_2^\mu k_2^\nu}{\left(k_1^2-m^2\right)^2\left(k_2^2-m^2\right)^2[(k_1-k_2)^2-m^2]} \tag{126}$$

and the finite part coming from this contribution is

$$\Gamma^2_{\rm fin} = \frac{i\lambda^4}{2}(I_1+2I_2+I_3) \tag{127}$$

with

$$I_1 = \int \frac{d^6k_1}{(2\pi)^6}\int \frac{d^6k_2}{(2\pi)^6}\frac{p^4}{\left(k_1^2-m^2\right)^3\left(k_2^2-m^2\right)^3[(k_1-k_2)^2-m^2]} \tag{128}$$

$$I_2 = \int \frac{d^6k_1}{(2\pi)^6}\int \frac{d^6k_2}{(2\pi)^6}\frac{-4p^4(p\cdot k_2)^3}{\left(k_1^2-m^2\right)^3\left(k_2^2-m^2\right)^4[(k_1-k_2)^2-m^2]} \tag{129}$$

$$I_3 = \int \frac{d^6k_1}{(2\pi)^6}\int \frac{d^6k_2}{(2\pi)^6}\frac{16(p\cdot k_2)^2(p\cdot k_2)^2}{\left(k_1^2-m^2\right)^4\left(k_2^2-m^2\right)^4[(k_1-k_2)^2-m^2]} \tag{130}$$

$$\Gamma_{\rm nonlocal} = \Gamma^3_{\rm fin} + \Gamma^{\rm div}_{\rm nonlocal}. \tag{131}$$

In terms of functions $\Theta^{(1)}(k_i,\,p,\,m^2)$ and $\Upsilon^{(1)}(k_i,\,m^2)$ we can write

$$\Gamma^{\rm div}_{\rm nonlocal} = \frac{(-i\lambda)^4(i)^5}{2}\int_\Lambda \frac{d^6k_1}{(2\pi)^6}\int_\Lambda \frac{d^6k_2}{(2\pi)^6}\left\{\Theta^{(1)}(k_1,\,p,\,m^2)\Upsilon^{(1)}(k_2,\,m^2) + \Theta^{(1)}(k_2,\,p,\,m^2)\Upsilon^{(1)}(k_1,\,m^2)\right\}. \tag{132}$$

Note that $\Upsilon^{(1)}$ is (the integrand of a) logarithmic divergence, which, in DR would give us $1/\epsilon$ and when multiplied by the remaining pieces of the amplitude would

produce the celebrated term $\ln p^2/\epsilon$ (Kaku, 1993). The (other) finite contributions are

$$\Gamma^1_{\text{fin}} = \frac{(-i\lambda)^4(i)^5}{2}\int\frac{d^6k_1}{(2\pi)^6}\int\frac{d^6k_2}{(2\pi)^6}\Theta^{(2)}(k_1, k_2, p, m^2) \tag{133}$$

and

$$\begin{aligned}\Gamma^3_{\text{fin}} = \frac{(-i\lambda)^4(i)^5}{2}&\int\frac{d^6k_1}{(2\pi)^6}\int\frac{d^6k_2}{(2\pi)^6}\\ &\times\left\{\Theta^{(1)}(k_1, p, m^2)\frac{2k_1\cdot k_2 - k_1^2}{\left(k_2^2-m^2\right)^3[(k_1-k_2)^2-m^2]}\right.\\ &+\Theta^{(1)}(k_2, p, m^2)\frac{2k_1\cdot k_2 - k_2^2}{\left(k_1^2-m^2\right)^3[(k_1-k_2)^2-m^2]}\\ &+\int\frac{d^6k_1}{(2\pi)^6}\int\frac{d^6k_2}{(2\pi)^6}R(k_1, k_2, \lambda)\\ &\times f^{\text{fin}}(p, k_1, m^2)[f_1(k_2, m^2)+ f_2(k_2, m^2)]\\ &+ f^{\text{fin}}(p, k_2, m^2)[f_1(k_1, m^2)+ f_2(k_1, m^2)]]\Big\}.\end{aligned} \tag{134}$$

Summarizing, the two-loop renormalization constants obtained are,

$$\begin{aligned}A^{(2)} = \frac{\lambda^4}{2}&\left[\frac{4}{6}g_{\mu\nu}I^{\mu\nu(2)}_{\log\,1}(m^2, \Lambda) - I^{(2)}_{\log\,1}(m^2, \Lambda)\right]\\ &+\lambda^2\left[\frac{4}{6}g_{\mu\nu}I^{\mu\nu(2)}_{\log\,2}(m^2, \lambda^2, \Lambda) - I^{(2)}_{\log\,2}(m^2, \lambda^2, \Lambda)\right]\end{aligned} \tag{135}$$

$$\delta^{(2)}m^2 = -\left[\frac{\lambda^4}{2}I^{(2)}_{\text{quad1}}(m^2, \Lambda) + \lambda^2 I^{(2)}_{\text{quad}\,2}(m^2, \lambda^2, \Lambda)\right] \tag{136}$$

and

$$B^{(2)} = \lambda^4\left[I^{(2)}_{\log\,1}(m^2, \Lambda) + I^{(2)}_{\log\,3}(m^2, \Lambda)\right] + 3\lambda^2 I^{(2)}_{\text{log2}}(m^2, \lambda^2, \Lambda) \tag{137}$$

where

$$I^{(2)}_{\log\,1}(m^2, \Lambda),\ I^{\mu\nu(2)}_{\log\,1}(m^2, \Lambda),\ I^{(2)}_{\log\,2}(m^2, \lambda^2, \Lambda),\ I^{\mu\nu(2)}_{\log\,2}(m^2, \lambda^2, \Lambda),\ I^{(2)}_{\log\,3}(m^2, \Lambda)$$

and

$$I^{(2)}_{\text{quad1}}(m^2, \Lambda),\quad I^{(2)}_{\text{quad2}}(m^2, \lambda^2, \Lambda)$$

are defined in Eqs. (56), (126), (78), (120), (91), (125), and (119), respectively.

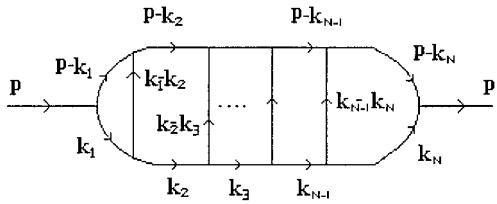

Fig. 9. The n-loop self-energy contribution $i\,\Sigma_1^{(n)}(p^2)$.

3.2.3. The N-Loop Order

Let us first consider the overlapping self-energy diagram of Fig. 9. It corresponds to the amplitude

$$i\,\Sigma_1^{(n)}(p^2) = (\Pi R)(\Pi f) \tag{138}$$

with

$$(\Pi R) = \frac{(-i\lambda^{2n})(i)^{3n-1}}{2}\left\{\prod_{j=1}^{n}\int_{\Lambda}\frac{d^6k_j}{(2\pi)^6}\frac{1}{k_j^2-m^2}\right\}Q(k_i, k_{i+1}, m^2). \tag{139}$$

The external momentum dependent part is

$$(\Pi f) = \prod_{j=1}^{n}\frac{1}{[(p-k_j)^2-m^2]}. \tag{140}$$

Using the technique we have, as usual

$$i\,T^2\Sigma_1^{(n)}(p^2) = \Gamma^1_{\text{fin}} + \Gamma_{\text{local}} + \Gamma_{\text{nonlocal}} \tag{141}$$

where

$$\Gamma^1_{\text{fin}} = \frac{(-i\lambda)^{2n}(i)^{3n-1}}{2}\prod_{j=1}^{n}\int\frac{d^6k_j}{(2\pi)^6}\Theta^{(n)}(k_1, k_2, \ldots k_n; p, m^2) \tag{142}$$

and

$$\begin{aligned}\Gamma_{\text{local}} &= \Gamma^2_{\text{fin}} + \Gamma^{\text{div}}_{\text{local}}\\ &= (\Pi R)\prod_{j=1}^{n}\{f^0(k_j, m^2) + f^1(k_j, m^2, p) + f^1(k_j, m^2, p)\}.\end{aligned} \tag{143}$$

In this case we have $\Gamma^2_{\text{fin}} \neq 0$. The counterterms characteristic of the *nth* order are identified in the equation

$$\Gamma^{\text{div}}_{\text{local}} = i\left(\delta_1^{(n)}m^2 - A_1^{(n)}p^2\right).$$

The nonlocal part is

$$\Gamma_{\text{nonlocal}} = \Gamma^3_{\text{fin}} + \Gamma^{\text{div}}_{\text{nonlocal}}. \tag{144}$$

where

$$\begin{aligned}\Gamma^{\text{div}}_{\text{nonlocal}} = {} & \frac{(-i\lambda)^{2n}(i)^{3n-1}}{2}\prod_{j=1}^{n}\int_{\Lambda}\frac{d^6k_j}{(2\pi)^6} \\ & \times\Bigg\{\sum_{a=1}^{n-1}\Theta^{(a)}(k_1,k_2,\ldots,k_a;p^2,m^2)\Upsilon^{(n-a)}(k_{a+1},\ldots,k_n,m^2) \\ & +\sum_{a=1}^{n-1}\Upsilon^{(n-a)}(k_1,k_2,\ldots,k_{n-a};m^2)\Theta^{(a)}(k_{n-a+1},\ldots,k_n;p^2,m^2) \\ & +\sum_{a,b=1}^{n-1}\Upsilon^{(b)}(k_1,k_2,\ldots,k_b;m^2)\Theta^{(a)}(k_{b+1},\ldots,k_a;p^2,m^2) \\ & \times\Upsilon^{(n-a-b)}(k_{a+1},\ldots,k_n;m^2)\Bigg\}\end{aligned} \tag{145}$$

From the above equation it becomes clear that the renormalization of the self-energy to *nth*-order requires all finite functions defined in previous self-energy diagrams (up to $(n-1)$th-order) as well as all the divergent contributions of the three-point functions also to the $(n-1)$th-order. We may associate a graphical representation to the equation above and, in this way, compare with the BPHZ results. The first term in Eq. (145) contains a sum of $n-1$ terms comprising a finite functions of the type Θ multiplied by the $n-a$ divergent vertex-type functions. The second term is symmetric to the first one (the vertex functions and functions Θ swap sides). Finally the last term contains vertex corrections to the left and to the right and finite functions in the middle. This can be best visualized in the graph which follows (Fig. 10).

Notice that in the present procedure no special treatment has been given to the overlapping divergencies or to the nested ones, both appearing in the self-energy. The reason is that the algebraic procedure produces only disjoint divergent contributions.

In order to complete the renormalization of this theory we will still consider two cases, both belonging to the second class defined previously. Firstly we consider a specific case where two-point functions explicitly appear as subdivergences (see Fig. 11) and the other is an amplitude containing an overlapping divergence diagram as substructure (Fig. 12). As we mentioned before, the total integral contains the two-point function substructures in factorized form. We therefore effect the renormalization of the internal propagators directly using the counterterms of order. In this way we immediately obtain $\bar{\Gamma}$. Let us first consider the case in

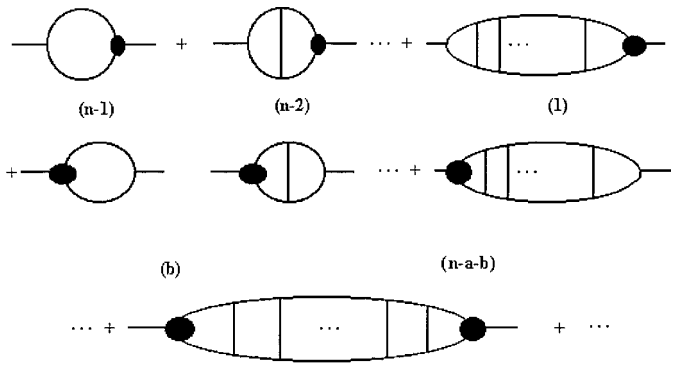

Fig. 10. Graphic representation of Eq. (145).

Fig. 11. This diagram contains subdiagrams involving nested two-point functions. Following the prescription which display the renormalized contributions of previous orders we get

$$\begin{aligned}\bar{\Gamma} &= i\,\bar{\Sigma}_{a,b=1}^{n-1} z_2^{(n)}(p^2) \\ &= (-i\lambda)^2 (i)^3 \int_\Lambda \frac{d^6 k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^2 \left[(p-k_n)^2 - m^2\right]} \\ &\quad \times i \left\{ T_{k_n}^2\, \bar{\Sigma}_2^{(n-1)}\left(k_n^2\right) + \delta_2^{(n-1)} m^2 - A_2^{(n-1)} k_n^2 \right\} \end{aligned} \tag{146}$$

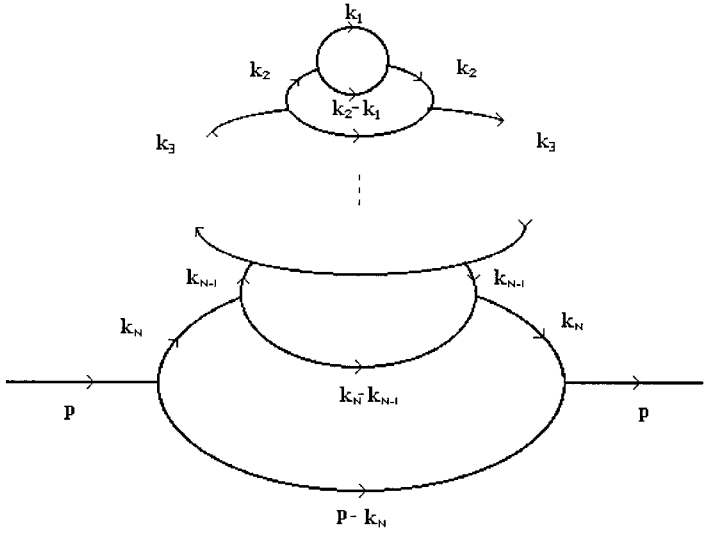

Fig. 11. The n-loop self-energy contribution $i\,\Sigma_2^{(n)}(p^2)$.

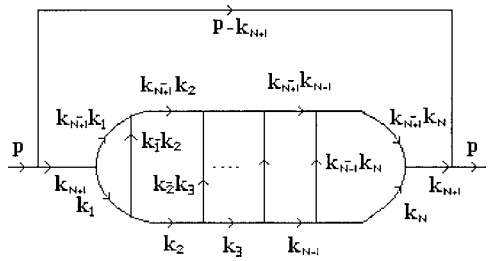

Fig. 12. The n-loop self-energy contribution $i\,\Sigma_3^{(n)}(p^2)$.

with

$$i\,\bar{\Sigma}_2^{(n-1)}\left(k_n^2\right) = (-i\lambda)^2(i)^3\int_\Lambda \frac{d^6k_{n-1}}{(2\pi)^6}\frac{1}{\left(k_{n-1}^2-m^2\right)^2\left[(k_n-k_{n-1})^2-m^2\right]} \times i\left\{T_{k_{n-1}}^2\,\bar{\Sigma}_2^{(n-2)}\left(k_{n-1}^2\right)+\delta_2^{(n-2)}m^2-A_2^{(n-2)}k_{n-1}^2\right\} \tag{147}$$

$$\vdots$$

$$i\,\bar{\Sigma}_2^{(2)}\left(k_3^2\right) = (-i\lambda)^2(i)^3\int_\Lambda \frac{d^6k_2}{(2\pi)^6}\frac{1}{\left(k_2^2-m^2\right)^2\left[(k_3-k_2)^2-m^2\right]} \times i\left\{T_{k_2}^2\,\Sigma^{(1)}\left(k_2^2\right)+\delta_1 m^2-A^{(1)}k_2^2\right\} \tag{148}$$

$$i\,\Sigma^{(1)}\left(k_2^2\right) = (-i\lambda)^2(i)^2\int_\Lambda \frac{d^6k_1}{(2\pi)^6}\frac{1}{\left(k_1^2-m^2\right)^2\left[(k_2-k_1)^2-m^2\right]}. \tag{149}$$

We can substitute the terms in brackets by renormalized function

$$\bar{\Sigma}_{2\mathrm{R}}^{(n-1)}\left(k_n^2\right) = \Gamma_{\mathrm{R}}^{(n-1)} = T_{k_n}^2\,\bar{\Sigma}_2^{(n-1)}\left(k_n^2\right)+\delta_2^{(n-1)}m^2A_2^{(n-1)}k_n^2. \tag{150}$$

In this example the two-point function counterterms can be obtained for any order $n > 1$ as

$$i\delta_2^{(n)}m^2 = (-i\lambda)^2(i)^3\int_\Lambda \frac{d^6k_n}{(2\pi)^6}\frac{1}{\left(k_n^2-m^2\right)^3}\left\{i\,\bar{\Sigma}_{2\mathrm{R}}^{(n-1)}\left(k_n^2\right)\right\} \tag{151}$$

$$iA_2^{(n)} = (-i\lambda)^2(i)^3\left\{\frac{4}{6}g^{\mu\nu}\int_\Lambda \frac{d^6k_n}{(2\pi)^6}\frac{k_\mu k_\nu}{\left(k_n^2-m^2\right)^5}\left\{i\,\bar{\Sigma}_{2\mathrm{R}}^{(n-1)}\left(k_n^2\right)\right\} - \int_\Lambda \frac{d^6k_n}{(2\pi)^6}\frac{1}{\left(k_n^2-m^2\right)^4}\left\{i\,\bar{\Sigma}_{2\mathrm{R}}^{(n-1)}\left(k_n^2\right)\right\}\right\}. \tag{152}$$

Finally we will consider the type 3 diagram (which contain a type 1 diagram) shown in Fig. 12. The corresponding amplitude reads

$$i\Sigma_3^{(n)}(p^2) = (-i\lambda)^2(i)^3 \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^2 [(p-k_n)^2 - m^2]} \left\{ i\Sigma_1^{(n-1)}\left(k_n^2\right)\right\}. \tag{153}$$

Note that since the structure $\sum_1^{(n-1)}(k_n^2)$ can be renormalized at $(n-1)$th order (see first example of order n), the *nth* order structure will also be renormalized by a $\Gamma^{\text{div}}_{\text{nonlocal}}$. Its counterterms have the same form as in the previous example. They are

$$i\delta_3^{(n)}m^2 = (-i\lambda)^2(i)^3 \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^3} \left\{ i\Sigma_{1\text{R}}^{(n-1)}\left(k_n^2\right)\right\} \tag{154}$$

$$iA_3^{(n)} = (-i\lambda)^2(i)^3 \left\{ \frac{4}{6} g^{\mu\nu} \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{k_\mu k_\nu}{\left(k_n^2 - m^2\right)^5} \left\{ i\Sigma_{1\text{R}}^{(n-1)}\left(k_n^2\right)\right\} - \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^4} \left\{ i\Sigma_{1\text{R}}^{(n-1)}\left(k_n^2\right)\right\}\right\}. \tag{155}$$

Note that in this case the three-point function subdiagrams have been renormalized together with the two-point subdiagram, since it is contained in the latter.

4. MOMENTUM ROUTING INDEPENDENCE

In the examples of the previous sections we have chosen the momentum routing in such a way as to obtain the simplest form for the final expressions. Of course, the counterterms so obtained must be independent of the particular routing one chooses. In order to exemplify this we consider the last example given (type 3 diagram). One of the possible choices for the momentum routing would be to arrange the labels in such a way that external momentum is present in an internal line of the diagram as follows

$$i\Sigma_3^{(n)}(p^2) = (-i\lambda)^2(i)^3 \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^2 [(p-k_n)^2 - m^2]^2} \left\{ i\Sigma_1^{(n-1)}((p-k_n)^2)\right\}. \tag{156}$$

and also,

$$i\Sigma_3^{(n)}(p^2) = (-i\lambda)^2(i)^3 \int_\Lambda \frac{d^6k_n}{(2\pi)^6} \frac{1}{\left(k_n^2 - m^2\right)^2 [(p-k_n)^2 - m^2]} \left\{ i\Sigma_1^{(n-1)}\left(k_n^2\right)\right\}. \tag{157}$$

These two labels must be equivalent, so that the amplitude is momentum routing independent as it should. Note that if the amplitude were finite, this could immediately be accomplished through a shift $p - k_n = k'_n$. However, since the amplitude is quadratically divergent, shifts are not allowed without the inclusion of surface terms. This point has been extensively discussed in our method (see Battistel, 1999; Battistel *et al.*, 1998; Scarpelli *et al.*, 2001) and a similar procedure can cure this problem in the present model. More difficult would be theories with gauge symmetries and work along this line is in progress. Note that in Dimensional Regularization the problem does not appear since shifts are always allowed.

5. CONCLUSION

We have considered (in the self-energy) all possible complications which usually appear in renormalization procedures: Overlapping divergences, nested divergencies, and disjoint ones, all in the same graph at n-loops. We have explicitly shown how these problems can be systematically resolved order by order within our technique in a simple example, devoid of symmetries abelian or not. However, several general aspects of the method can be learned already from this simple case: There will always be a divergent (local) order dependent contribution. Also, there will always be a finite contribution composed by the product of all finite parts of f_j's. These two structures (divergent and finite) are typical of the *nth* order and pose no problem for renormalization.

As we have seen in the examples given, the identities we use in the integrand leave us then with crossed products of divergent and finite contributions. All possible combinations will appear and all of them can either be recognized as structures (finite or divergent) already encountered in lower order amplitudes or they will give a finite contribution.

In summary we have presented a completely algebraic method of perturbative renormalization. All counterterms appear automatically, and present a close correspondence to BPHZ where graphical representations are essential. We feel that if our method also works for abelian and nonabelian theories this could be a major advance. At the one-loop level we can preserve gauge symmetry if use is made of relations involving divergent integrals of the same degree of divergence (Battistel, 1999; Battistel *et al.*, 1998; Scarpelli *et al.*, 2001). The difference among those integrals is the source of both ambiguities and symmetry violations. The results we got at one loop are encouraging. We are working on the application of this method at the two-loop level, the Quantum Electrodynamic (QED).

We see that the application of this method leads to a relatively simple renormalization procedure. There is no need for a graphic representation of the relevant contributions. When a diagram has divergent subdiagrams the subdivergences need not be previously identified because they will appear in an algebraic way. The procedure displays all relevant subdivergences. Investigations of advantages of the

present method (if at all!) over the existing ones remain as our main research interest right now.

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